Pumping Lemma for CFLs

The Big Picture

- CFL $\Longleftrightarrow$ CFG $\Longleftrightarrow$ PDA
- Every regular languages is a CFL.

Non-Context-Free Languages

- Property of CFL: strings longer than *pumping length* can be pumped.
- **Pumping lemma for CFL**

  Let L be context-free. Then $\exists$ pumping length p such that $\forall s \in L$ with $|s| \geq p$, s can be divided into five pieces $s = uvxyz$ satisfying

  1. $|vxy| \leq p$
  2. $|vy| \geq 1$
  3. $\forall i \geq 0, uv^ixy^iz \in L$

- Proof:
  - Let G be a CFG that generates L.
  - In G, long string $\Longrightarrow$ tall parse tree

    $\Longrightarrow$ repeating some variable in derivation

Proof

- Let b : maximum # of symbols in the RHS of a rule. Assume $b \geq 2$.
- Let $|V|$: # of variables in G.
- Set pumping length $p = b^{|V|+2}$.
- For some $s \in L$, $|s| \geq p$: (pump s)
  - Let τ — a parse tree for s with smallest number of nodes.
  - $\Rightarrow$ height of $\tau \geq |V| + 2$
    $\Rightarrow$ length of the longest path of $\tau \geq |V| + 2$
    $\Rightarrow$ # of variables on the path $\geq |V| + 1$
    $\Rightarrow$ some variable R appears more than once on the path.
  - Divide s into $uvxyz$ based on R.

Using Pumping Lemma to Prove Languages not Context Free

- Pumping lemma specifies a property that all CFL have.
- If a language L doesn't have this property, then L is not CFL.
- Using pumping lemma to prove non-context-free
  - Assume L is a CFL and obtain a contradiction using conditions in pumping lemma:
    Choose a long string $s \in$ L.
    Show that s cannot be pumped.
- Examples
  - $L = \{a^n b^n c^n | n \geq 0\}$ not context free
  - $L = \{a^i b^j c^k | k \geq j \geq i \geq 0\}$ not context free
  - $L = \{ww | ww \in \{0, 1\}^*\}$ not context free