NP-Completeness

Polynomial Time Reductions

- Definition: Language A is *polynomial time reducible* to language B, $A \leq_p B$, if $\exists$ a polynomial time computable function f, where for every w,

$$w \in A \Longleftrightarrow f(w) \in B.$$

- Theorem: $A \leq_p B$ and $B \in$ P, then $A \in$ P
- Transitivity: $A \leq_p B$ and $B \leq_p B$, then $A \leq_p C$.
- Examples: HC $\leq_p$ TSP, 3SAT $\leq_p$ VC

Definition of NP-Completeness

- NP-Complete:

  A language L is NP-complete if

  (1) L $\in$ NP;

  (2) $\forall A \in$ NP, A $\leq_p$ L.

  (If (2) holds, but not necessary (1), then L is NP-hard.)

- Theorem:

  If L is NP-complete and L $\in$ P, then P $=$ NP.

- Theorem:

  If B is NP-complete and B $\leq_p$ C for C in NP, then C is NP-complete.

  (The first NP-Complete problem is difficult to establish. Once we have one, we can more easily obtain others.)

- Examples: 3SAT, CLIQUE, TSP, VC, HC, $\cdots$

Space Complexity

- Space complexity of TM M: the maximum number of tape cells that M scans on any input of length M.
- Space appears to be more powerful than time.
- SPACE$(f(n)) = \{L \mid L$ is decided by a $O(f(n))$ space deterministic TM $\}$

  e.g. SAT $\in$ SPACE(n)
- NSPACE$(f(n)) = \{L \mid L$ is decided by a $O(f(n))$ space nondeterministic TM $\}$
- Savitch's Theorem: For any function $f : N \longrightarrow N$, where $f(n) \geq n$, NSPACE$(f(n)) \subseteq$ SPACE$(f^2(n))$

The Class PSPACE

- PSPACE $= \cup_{k \geq 0}$ SPACE(n^k)
- NPSPACE $=$ PSPACE
- The BIG Picture

P $\subseteq$ NP $\subseteq$ PSPACE $=$ NPSPACE $\subseteq$ EXPTIME